- Lecture 11/10/21 - Section ??
- last time: Line integrals
- FTLI: If f is a function with cts partial derivatives and C is a smooth curve parameterized by $\vec{r}(t)$ on $[a,b]$, then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Ex: Compute $\int_C \vec{v} \cdot dr$ for $\vec{v} = \langle \sin(y), x\cos(y) + \cos(z), -y\sin(z) \rangle$ for C parameterized by $\vec{r}(t) = \langle \sin(t), t, 2t \rangle$ on $[0, \frac{\pi}{2}]$

Sol: check if FTLI holds:

① check if $\vec{v}$ is conservative (does it satisfy Clairaut's thm?):

$$\frac{\partial}{\partial y}[V_x] = \frac{\partial}{\partial y}[\sin(y)] = \cos(y)$$

$$\frac{\partial}{\partial z}[V_x] = \frac{\partial}{\partial z}[\sin(y)] = 0$$

$$\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial x}[x\cos(y) + \cos(z)] = \cos(y)$$

$$\frac{\partial}{\partial z}[V_y] = \frac{\partial}{\partial z}[x\cos(y) + \cos(z)] = -\sin(z)$$

$$\frac{\partial}{\partial x}[V_z] = \frac{\partial}{\partial x}[-y\sin(z)] = 0$$

$$\frac{\partial}{\partial y}[V_z] = \frac{\partial}{\partial y}[-y\sin(z)] = -\sin(z)$$

Partials match so Clairaut's thm and FTLI both hold

→

② compute potential function:

$\frac{\partial f}{\partial x} = \sin(y)$, $\frac{\partial f}{\partial x} = x\cos(y) + \cos(z)$, $\frac{\partial f}{\partial z} = -y\sin(z)$

- Now $f(x,y,z) = \int \frac{\partial f}{\partial x}\, dx = \int \sin(y)\, dx = \underline{x\sin(y) + C(y,z)}$,

- and $x\cos(y) + \cos(z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\left[x\sin(y) + C(y,z)\right] = x\cos(y) + \frac{\partial C}{\partial y}$

solve for $\frac{\partial C}{\partial y}$: $\frac{\partial C}{\partial y} = x\cos(y) + \cos(z) - x\cos(y) = \cos(z)$

- Hence: $C(y,z) = \int \frac{\partial C}{\partial y}\, dy = \int \cos(z)\, dy = y\cos(z) + D(z)$

- Now: $f(x,y,z) = x\sin(y) + C(y,z) = x\sin(y) + y\cos(z) + D(z)$

$\therefore -y\sin(z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x\sin(y) + y\cos(z)\ D(z)\right]$

$-y\sin(z) = -y\sin(z) + D'(z)$

$0 = D'(z)$

$\therefore D(z) = E$ where E is some constant

we chose $E = 0$

$\therefore f(x,y,z) = x\sin(y) + y\cos(z) + 0$

③ Now Apply FTLI:

$\int_C \vec{v} \cdot dr = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$\cdot r(b) = r(\frac{\pi}{2}) = \langle \sin(\frac{\pi}{2}), \frac{\pi}{2}, 2(\frac{\pi}{2}) \rangle = \langle 1, \frac{\pi}{2}, \pi \rangle$

$\underbrace{\cdot r(a) = r(0) = \langle \sin(0), 0, 2(0) \rangle = \langle 0, 0, 0 \rangle}$ →

these came from $\vec{r}(t)$ which was given, we just plugged in our endpoints $[0, \frac{\pi}{2}]$.

we computed this

$$f(x,y,z) = x\sin(y) + y\cos(z)$$

- Lastly: $\int_C \vec{v}\cdot d\vec{r} = f(1, \frac{\pi}{2}, \pi) - f(0,0,0)$

$$= f(1, \tfrac{\pi}{2}, \pi) - f(0,0,0)$$

$$= \left(1\left(\sin(\tfrac{\pi}{2})\right) + \tfrac{\pi}{2}\cos(\pi)\right) - \left(0\sin(0) + 0\cos(0)\right)$$

$$= 1 + \tfrac{\pi}{2}(-1) - 0$$

$$= 1 - \tfrac{\pi}{2}$$

- <u>Recall</u>: changing orientation of the curve (the direction we go on it) negates the curve

i.e. $\int_{-C} \vec{v}\cdot d\vec{r} = -\int_C \vec{v}\cdot d\vec{r}$

- <u>Independence of path for line integrals of conservative vector fields.</u>

- Prop: given a conservative v.f. $\vec{v}$ and 2-points α, β; we have $\int_C \vec{v}\cdot d\vec{r} = f(\beta) - f(\alpha)$ for every curve from α to β

- Prop: A v-field defined on some open region R is conservative iff for all points α, β in R and all curves C, D from α to β we have $\int_C \vec{v}\cdot d\vec{r} = \int_D \vec{v}\cdot d\vec{r}$
i.e. $\vec{v}$ is conservative precisely when it satisfies independence of paths.

→

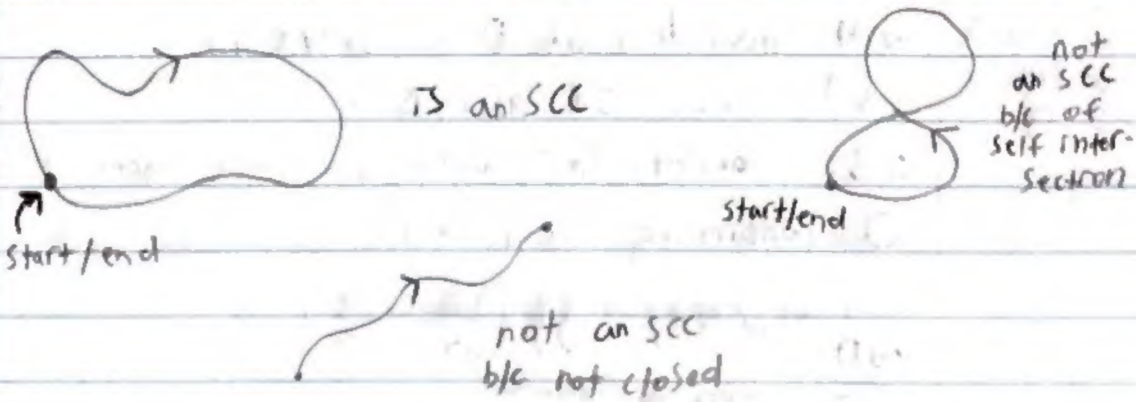
claim: If $\vec{v}$ satisfies independence of paths, then define a function $f = \int_{\alpha}^{\vec{x}} \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot d\vec{r}$ (for any curve from α to $\vec{x}$ where α is fixed in advance)

- The function f makes sense b/c $\int_C \vec{v} \cdot d\vec{r}$ is independent of C.

- what remains follows from the chain rule and the FTC (exercise)

<u>Defn</u>: A simple closed curve is a curve w/o self intersection which starts and ends at the same point.

Pictures:



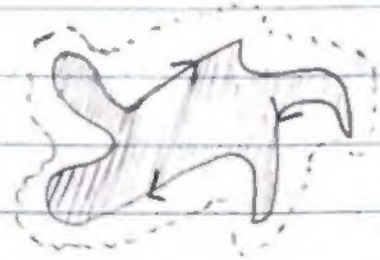
Prop: A v.f. defined in open Region R is conservative iff for all simple closed curves C we have $\vec{v} \cdot d\vec{r} = 0$

$\longrightarrow$

§16.4: Greens Theorem

- Idea: we want to connect some special line integrals to double integrals.

Picture:



- Idea: turn a line integral over a region cut out by an scc into a double $\int$.

- Suppose we have D, a closed Region in $\mathbb{R}^2$ with boundary of D a simple, piecewise-smooth, closed curve. If $P(x,y)$ and $Q(x,y)$ have cts. partial derivatives on some open region O containing D, then

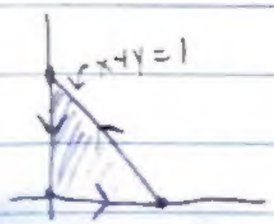
∂D means boundary of D

$$\int_{\partial D} P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

← positively oriented

[EX] compute $\int_C x^4\,dx + xy\,dy$ for C the positively oriented curve around the triangle w/ vertices $(0,0)$, $(1,0)$, $(0,1)$

Picture:



Parameterize the region:

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

and $\partial D = C$

→

$\therefore$ by Greens thm we compute:

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[x^4]\right) dA$$

$$= \iint_D (y - 0)\,dA = \iint_D (y)\,dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} y\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{1}{2}y^2 \Big|_0^{1-x} dx = \frac{1}{2}\int_{x=0}^{1} (1-x)^2\,dx$$

let $u = 1-x$
$du = -dx$

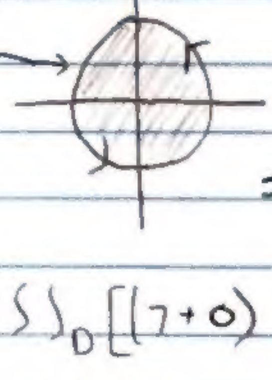
$$= \frac{1}{2}\cdot\frac{1}{3}\left[(1-x)^3\right]_{x=0}^{1} = -\frac{1}{6}\left((1-1)^3 - (1-0)^3\right)$$

$$= -\frac{1}{6}(-1) = \frac{1}{6}$$

Ex] compute $\int_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$
for C the positive oriented curve around circle $x^2+y^2=16$

picture:

D



Solution: By green's thm:

$$\int_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$$

$$= \iint_D \frac{\partial}{\partial x}\left[7x + \sqrt{y^4+1}\right] - \frac{\partial}{\partial y}\left[3y - e^{\sin(x)}\right]$$

$$= \iint_D \left[(7+0) - (3-0)\right]$$

$$\iint_D 4\,dA = 4\iint_D dA = 4\,\text{Area}(D) = 4\pi(4)^2 = 64\pi$$